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SOME PROPERTIES OF HARMONIC UNIVALENT FUNCTIONS USING Q-CALCULUS AND ERROR FUNCTION

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Abstract: This paper investigates a subclass of harmonic univalent functions using q-calculus and error functions. We determine the necessary condition for the complex valued harmonic, univalent and sense-preserving function f to be the class of harmonic error functions. Further, the necessary and sufficient conditions for functions f to be a member of the subfamily and its characterization, followed by extreme points of the class are determined. Additionally, we prove that the class is closed under convex combinations, indicating that linear combinations of functions within the class also belong to the class. These estimates illuminate the growth and deformation of functions within the unit disk. These findings enhance our understanding of the geometric and analytic properties of harmonic error functions and their significance in the field of geometric function theory.

Keywords and Phrases: Harmonic function, holomorphic function, univalent function, q-calculus, convolution, error function.

2020 Mathematics Subject Classification: 30C45, 30C50, 30C80.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply-connected domain D if both u and v are real and harmonic in D. If h and g are analytic in D, then f can be expressed in the form:

$$f = h + \overline{g}, \tag{1.1}$$

where h is analytic part and g is the co-analytic part of f. Consider the complex valued harmonic function $f = h + \overline{g}$ with $h(0) = h'(0) - 1 = 0$ with the following power series expansion:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.2}$$

Let S_H denote the class of all complex valued harmonic , univalent and sense-preserving function f of the function (1.1) defined in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ for $f(0) = g(0) = h'(0) - 1 = 0$. Obviously, the sense-preserving property implies $|b_1| < 1$(see [2], [8], [10]). Thus each function $f \in S_H$ can be written in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (z \in \Delta). \tag{1.3}$$

Clunie and Sheil-Small [6] proved that the class S_H is not compact and the necessary and sufficient conditions for f to be locally univalent and sense-preserving in any simply connected domain Δ is that

$$|k| = \left|\frac{g'}{h'}\right| < 1$$

where k is the second dilatation of f and $h'(z) \neq 0$. If the co-analytic part $g \equiv 0$, then the class S_H reduces to the class $\mathcal{S}$ of normalized analytic univalent functions. For more information go through the following articles see ([5], [7], [11], [14], [15]). Now we recall the notion of q-operator or q-difference operator that was used in geometric function theory, quantum physics and operator theory. For $0 < q < 1$, we define the q-integer $[n]_q$ by

$$[n]_q = \frac{1 - q^n}{1 - q} \quad (n = 1, 2, 3...).$$

Note that if $q \to 1^-$, the $[n]_q \to n$. The q-analogue of Noor integral operator is defined as follows:

$$\Im_q{}^{\mu} f(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n$$

(see [4]) where

$$\psi_{n-1} = \frac{[n, q]!}{[\mu + 1, q]_{n-1}}$$

and

$$\Im_q{}^{\mu} f(z) = \Im_q{}^{\mu} h(z) + \overline{\Im_q{}^{\mu} g(z)}$$

Abramowitz and Stegun (see [1]) defined error function as

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} z^{2n+1} = \frac{2}{\sqrt{\pi}} \left[z - \frac{z^3}{3} + \frac{z^5}{52!} - \frac{z^7}{73!} + ... \right]$$

The normalized analytic error function is defined as follows:

$$\begin{aligned} E_r f(z) &= \frac{\sqrt{\pi z}}{2} erf(\sqrt{z}) \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n \end{aligned}$$

(see [13]) and defined the family;

$$\begin{aligned} \mathcal{E} = f * E_r f(z) &= \{ \mathcal{E}Rf : \mathcal{E}Rf(z) = (f \star E_r(f))(z) \} \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} a_n z^n, f \in \mathcal{A} \}. \end{aligned}$$

Error functions will be analytic everywhere with respect to Taylor series. It has no singularities in the complex plane. The Taylor expansion of entire function always convergent. A good amount of literature exists in the field of harmonic univalent functions and $q-$calculus. For details, see ([3], [8], [9], [12]).
Let H and G be analytic functions in Δ and consider

$$\Im_q{}^{\mu} \mathcal{E}RF = \Im_q{}^{\mu} \mathcal{E}RH + \overline{\Im_q{}^{\mu} \mathcal{E}RG}$$

where

$$\Im_q{}^{\mu} \mathcal{E}RH(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} a_n z^n$$

and

$$\Im_q{}^{\mu} \mathcal{E}RG(z) = \sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} b_n z^n.$$

Let us consider the family of harmonic error functions f as $\mathcal{E}N_H(\mu, \nu, \lambda)$ and of the form (1.3) such that

$$\Re \left\{ \frac{z(\Im_q{}^{\mu} \mathcal{E}Rh(z))_z - \bar{z}(\Im_q{}^{\mu} \mathcal{E}Rg(z))_{\bar{z}}}{(1-\lambda)z + \lambda \Im_q{}^{\mu} \mathcal{E}Rf(z)} \right\} \geq \nu \tag{1.4}$$

where $(0 \leq \mu < 1,\ \lambda \in \mathbb{N} \cup \{0\}\ \ \&\ \ 0 < q < 1)$. Also let

$$\mathcal{E}N_{\overline{H}}(\mu, \lambda, \nu) \equiv \mathcal{E}N_H(\mu, \lambda, \mu) \cap \overline{H}$$

where $\overline{H}$ is the subfamily of H consisting of harmonic functions of the form;

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n;\ a_n \geq 0, b_n \geq 0. \tag{1.5}$$

Let $\mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$ represent the subfamily of $\mathcal{E}N_H(\mu, \nu, \lambda)$ consists of harmonic functions $f = h + \overline{g}$ such that h, g are of the form

$$h(z) = z + \sum_{n=2}^{\infty} (-1)^n |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} |b_n| z^n \tag{1.6}$$

2. Main Result

The arguments present to prove the theorems in this section are similar to those given in [9] and [8]. For a special case of the results presented in this section, see [8].

Theorem 2.1. *If a function f in the form (1.3) fulfills*

$$\sum_{n=1}^{\infty} \psi_{n-1} \frac{(n - \lambda\nu)|a_n| + (n + \lambda\nu)|b_n|}{(1 - \nu)(2n - 1)(n - 1)!} \leq 2, \tag{2.1}$$

then $f \in \mathcal{E}N_H(\mu, \nu, \lambda)$, harmonic univalent and sense preserving.

Proof. Let suppose that,

$$\begin{aligned}
A(z) &= z(\Im_q{}^{\mu}\mathcal{E}Rh(z))_z - \bar{z}(\Im_q{}^{\mu}\mathcal{E}Rg(z))_{\bar{z}} \\
&= z\left(z + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1}}{((2n-1)(n-1)!} a_n z^n\right)_z - \bar{z}\left(\sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n\right)_{\bar{z}} \\
&= z\left(1 + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^{n-1}\right) - \bar{z}\left(\sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^{n-1}\right) \\
&= z + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n
\end{aligned}$$

and

$$\begin{aligned}
B(z) &= (1-\lambda)z + \lambda \Im_q{}^{\mu}\mathcal{E}Rf(z) \\
&= (1-\lambda)z + \lambda\left(z + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \right) \\
&= z + \sum_{n=2}^{\infty} \lambda\psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \lambda\psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n.
\end{aligned}$$

We use the following statement that was used in [8]. We use the following statement that was used in [8]. $\Re\{w\} \geq \nu$ iff $|(1-\nu)+w| \geq |(1+\nu)-w|$.
It suffices to show that,

$$|A(z) + (1-\nu)B(z)| - |A(z) - (1+\nu)B(z)| \geq 0.$$

$$\begin{aligned}
&|A(z) + (1-\nu)B(z)| - |A(z) - (1+\nu)B(z)| \\
&= \Bigg| z + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \\
&+ (1-\nu)\left(z + \sum_{n=2}^{\infty} \lambda\psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \lambda\psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \right) \Bigg| \\
&+ \Bigg| z + \sum_{n=2}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \\
&- (1+\nu)\left(z + \sum_{n=2}^{\infty} \lambda\psi_{n-1} \frac{(-1)^{n-1} n}{((2n-1)(n-1)!} a_n z^n - \sum_{n=1}^{\infty} \lambda\psi_{n-1} \frac{(-1)^{n-1} n}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \right) \Bigg| \\
&= \Bigg| (2-\nu)z + \sum_{n=2}^{\infty} (n + (1-\nu)\lambda)\psi_{n-1} \frac{(-1)^{n-1}}{((2n-1)(n-1)!} a_n z^n \\
&\qquad - \sum_{n=1}^{\infty} (n - (1-\nu)\lambda)\psi_{n-1} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \\
&\qquad + \Bigg| -\nu z + \sum_{n=2}^{\infty} (n - (1+\nu)\lambda)\psi_{n-1} \frac{(-1)^{n-1}}{((2n-1)(n-1)!} a_n z^n \\
&\qquad\quad - \sum_{n=1}^{\infty} (n + (1+\nu)\lambda)\psi_{n-1} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} \bar{b_n} \bar{z}^n \Bigg| \\
&\geq (2-\nu)|z| - \sum_{n=2}^{\infty} (n + (1-\nu)\lambda)\psi_{n-1} \frac{(-1)^{n-1}}{((2n-1)(n-1)!} a_n z^n
\end{aligned}$$

$$-\sum_{n=1}^{\infty}(n-(1-\nu)\lambda)\psi_{n-1}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}\bar{b_n}\bar{z}^n$$

$$-\nu z+\sum_{n=2}^{\infty}(n-(1+\nu)\lambda)\psi_{n-1}\frac{(-1)^{n-1}}{((2n-1)(n-1)!}a_n z^n$$

$$-\sum_{n=1}^{\infty}(n+(1+\nu)\lambda)\psi_{n-1}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}\bar{b_n}\bar{z}^n$$

$$=2(1-\nu)|z|\left(1-\sum_{n=2}^{\infty}\frac{(n-\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}|a_n||z|^{n-1}\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{(n+\lambda\nu)}{(2n-1)(n-1)!}\psi_{n-1}|\bar{b_n}||\bar{z}|^{n-1}\right)\geq 0.$$

where

$$\sum_{n=1}^{\infty}\psi_{n-1}\frac{(n-\lambda\nu)|a_n|+(n+\lambda\nu)|b_n|}{(1-\nu)(2n-1)(n-1)!}\leq 2.$$

To establish that f is univalent in Δ, we observe that if $g(z)\equiv 0$, the function f is analytic, and its univalence is guaranteed by its starlikeness. In the case where $g(z)\not\equiv 0$, we must demonstrate that $f(z_1)\neq f(z_2)$ for any distinct pair $z_1, z_2\in\Delta$.

Let z_1 and z_2 be distinct points in Δ. Since Δ is a convex, simply connected domain, the line segment joining them, defined by

$$z(t)=(1-t)z_1+tz_2,\quad 0\leq t\leq 1 \tag{2.2}$$

is contained entirely within Δ. So we can write

$$f(z_2)-f(z_1)=\int_0^1\left[(z_2-z_1)h'(z(t))+\overline{(z_2-z_1)g'(z(t))}\right]dt.$$

By dividing through by $z_2-z_1\neq 0$ and considering only the real components, it follows that

$$\begin{aligned}\mathrm{Re}\frac{f(z_2)-f(z_1)}{z_2-z_1}&=\int_0^1\mathrm{Re}\left[h'(z(t))+\frac{\overline{z_2-z_1}}{z_2-z_1}\overline{g'(z(t))}\right]dt\\&>\int_0^1\left[\mathrm{Re}\ h'(z(t))-|g'(z(t))|\right]dt.\end{aligned} \tag{2.3}$$

In contrast

$$\begin{aligned}\text{Re } h'(z) - |g'(z)| &\geq \text{Re } h'(z) - \sum_{n=1}^{\infty} n|b_n| \\ &\geq 1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n| \\ &\geq 1 - \sum_{n=2}^{\infty} \frac{(n-\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}|a_n| - \sum_{n=1}^{\infty} \frac{(n+\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}|b_n| \\ &\geq 0, \quad \text{by using (2.1).}\end{aligned}$$

Taken together with Inequality (2.3), this implies that f is univalent.

Next we need to check for sense preserving nature of f that is $|h'(z)| \geq |g'(z)|$ in the following.

$$\begin{aligned}|h'(z)| &= \left|1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right| \\ &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \\ &\geq 1 - \sum_{n=2}^{\infty} \frac{(n-\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}|a_n| \\ &\geq \sum_{n=1}^{\infty} \frac{(n+\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}|b_n| \\ &> \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \\ &\geq |g'(z)|.\end{aligned}$$

So, f is sense preserving in $\mathbb{D}$. Hence we conclude that $f \in \mathcal{E}N_H(\mu, \nu, \lambda)$.

In the following theorem the necessary and sufficient conditions for function f to be in subclass $\mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$ deduced in view of Theorem 2.1 is established.

Theorem 2.2. *$f \in \mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$ of the form (1.5) iff the the condition (2.1) holds true.*

Proof. In view of Theorem 2.1, we must show that each function $f \in \mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$ fulfills the inequality (2.1). We deduce the condition into following

$$\Re\left(\frac{2(1-\nu)z - \sum_{n=2}^{\infty} \frac{(n-\lambda\nu)\psi_{n-1}}{(2n-1)(n-1)!}|a_n|z^n - \sum_{n=1}^{\infty} \frac{(n+\lambda\nu)\psi_{n-1}}{(2n-1)(n-1)!}|\bar{b_n}|\bar{z}^n}{z - \sum_{n=2}^{\infty} \lambda\psi_{n-1}\frac{1}{(2n-1)(n-1)!}|a_n|z^n + \sum_{n=1}^{\infty} \lambda\psi_{n-1}\frac{1}{(2n-1)(n-1)!}|\bar{b_n}|\bar{z}^n}\right) \geq 0.$$

By taking $z = r$ $(0 \le r < 1)$ in the above expression, we must get the following

$$\frac{2(1-\nu) - \sum_{n=2}^{\infty} \frac{(n-\lambda\nu)\psi_{n-1}}{(2n-1)(n-1)!}|a_n|r^{n-1} - \sum_{n=1}^{\infty} \frac{(n+\lambda\nu)\psi_{n-1}}{(2n-1)(n-1)!}|\bar{b_n}|r^{n-1}}{1 - \sum_{n=2}^{\infty} \lambda\psi_{n-1}\frac{1}{(2n-1)(n-1)!}|a_n|r^{n-1} + \sum_{n=1}^{\infty} \lambda\psi_{n-1}\frac{1}{(2n-1)(n-1)!}|\bar{b_n}|r^{n-1}} \ge 0.$$

The denominator of the LHS cannot vanish for $r \in (0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in (0, 1)$. Thus, the proof of Theorem 2.2 is completed.

Distortion bounds for functions in $\mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$ are given below, yielding the necessary covering results.

Theorem 2.3. *If $f \in \mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$, then*

$$(1-|b_1|)r - \frac{1}{\psi_1}\left(\frac{1-\nu}{2-\lambda\nu} - \frac{1+\nu}{2-\lambda\nu}|b_1|\right)r^2 \le |f(z)| \le (1+|b_1|)r - \frac{1}{\psi_1}\left(\frac{1-\nu}{2-\lambda\nu} - \frac{1+\nu}{2-\lambda\nu}|b_1|\right)r^2$$

Proof. If $f \in \mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$, we obtain

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n \right| \\
&\le (1+|b_1|)|z| + \sum_{n=2}^{\infty}(a_n + b_n)|z|^n \\
&= (1+|b_1|)r + \sum_{n=2}^{\infty}(a_n + b_n)r^n \\
&\le (1+|b_1|)r + \sum_{n=2}^{\infty}(a_n + b_n)r^2 \\
&\le (1+|b_1|)r + \frac{3(1-\nu)}{(2-\lambda\nu)\psi_1}\sum_{n=2}^{\infty}\left(\frac{(2-\lambda\nu)\psi_1}{(1-\nu)}a_n + \frac{(2-\lambda\nu)\psi_1}{(1-\nu)}|b_n|\right)r^2 \\
&\le (1+|b_1|)r + \frac{3(1-\nu)}{(2-\lambda\nu)\psi_1}\left(1 - \frac{1+\nu}{1-\nu}b_1\right)r^2 \\
&= (1+|b_1|)r + \frac{1}{\psi_1}\left(\frac{3(1-\nu)}{2-\lambda\nu} - \frac{3(1+\nu)}{2-\lambda\nu}|b_1|\right)r^2 \\
&\le (1+|b_1|)r - \frac{1}{\psi_1}\left(\frac{1-\nu}{2-\lambda\nu} - \frac{1+\nu}{2-\lambda\nu}|b_1|\right)r^2
\end{aligned}$$

Similarly,

$$\begin{aligned}|f(z)| &\geq (1-|b_1|)|z| - \sum_{n=2}^{\infty}(a_n+b_n)|z|^n \\ &= (1-|b_1|)r - \sum_{n=2}^{\infty}(a_n+b_n)r^n \\ &\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(a_n+b_n)r^2 \\ &\geq (1-|b_1|)r - \frac{3(1-\nu)}{(2-\lambda\nu)\psi_1}\sum_{n=2}^{\infty}\left(\frac{(2-\lambda\nu)\psi_1}{(1-\nu)}a_n + \frac{(2-\lambda\nu)\psi_1}{(1-\nu)}|b_n|\right)r^2 \\ &\geq (1-|b_1|)r - \frac{3(1-\nu)}{(2-\lambda\nu)\psi_1}\left(1-\frac{1+\nu}{1-\nu}|b_1|\right)r^2 \\ &= (1-|b_1|)r - \frac{1}{\psi_1}\left(\frac{3(1-\nu)}{2-\lambda\nu} - \frac{3(1+\nu)}{2-\lambda\nu}|b_1|\right)r^2.\end{aligned}$$

Hence we get our desired result.

We now turn our attention to determining the extreme points for the set $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$.

Theorem 2.4. *The function f is in the closed convex hull of $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ iff f is in the form*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z))$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{(1-\nu)(2n-1)(n-1)!}{(n-\lambda\nu)\psi_{n-1}}z^n,$$

$$g_n(z) = z + \frac{(1-\nu)(2n-1)(n-1)!}{(n+\lambda\nu)\psi_{n-1}}\bar{z}^n,$$

$$\sum_{n=1}^{\infty}(X_n+Y_n) = 1,\ X_n \geq 0,\ Y_n \geq 0.$$

Proof. Consider

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z))$$

$$= \sum_{n=1}^{\infty}(X_n+Y_n)z - \sum_{n=2}^{\infty}\frac{(1-\nu)(2n-1)(n-1)!}{(n-\lambda\nu)\psi_{n-1}}X_n z^n + \sum_{n=2}^{\infty}\frac{(1-\nu)(2n-1)(n-1)!}{(n+\lambda\nu)\psi_{n-1}}Y_n\bar{z}^n.$$

Then,

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{(1-\nu)(2n-1)(n-1)!}{(n-\lambda\nu)\psi_{n-1}}&\left(\frac{(n-\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}X_n\right)\\&+\sum_{n=2}^{\infty}\frac{(1-\nu)(2n-1)(n-1)!}{(n+\lambda\nu)\psi_{n-1}}\left(\frac{(n+\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}Y_n\right)\\&=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\\&\leq 1.\end{aligned}$$

and so f is in the closed convex hull of $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$.
Conversely, if $f\in N_{\overline{H}}(\mu,\nu,\lambda)$, then

$$|a_n|\leq\frac{(n-\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}\text{ and }|b_n|\leq\frac{(n+\lambda\nu)\psi_{n-1}}{(1-\nu)(2n-1)(n-1)!}.$$

$$X_n=\frac{(1-\nu)(2n-1)(n-1)!}{(n-\lambda\nu)\psi_{n-1}}|a_n|,n\geq 2$$

$$Y_n=\frac{(1-\nu)(2n-1)(n-1)!}{(n+\lambda\nu)\psi_{n-1}}|a_n|,n\geq 1$$

and

$$X_1=1-\left(\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n\right);X_1\geq 0.$$

Hence $f(z)=\sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))$.

In the following result we have established closure behavior of convex combinations of functions in the class $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ with the help of the result (2.1) in Theorem 2.1.

Theorem 2.5. *The class $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ is closed under convex combinations.*
Proof. Let $f_i\in\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ then

$$f_i(z)=z+\sum_{n=2}^{\infty}(-1)^n|a_{n_i}|z^n+\sum_{n=1}^{\infty}(-1)^{n-1}|b_{n_i}|\bar{z}^n,\quad(i=1,2,3...).$$

Then by (2.1), we have

$$\sum_{n=2}^{\infty}\frac{(n-\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}|a_{n_i}|+\sum_{n=1}^{\infty}\frac{(n+\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}|b_{n_i}|\leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1 \quad (0 \leq t_i \leq 1)$

$$\sum_{i=1}^{\infty} f_i(z) = z + \sum_{n=2}^{\infty}(-1)^n\left(\sum_{n=1}^{\infty} t_i|a_{n_i}|\right)z^n + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{n=1}^{\infty} t_i|b_{n_i}|\right)\bar{z}^n$$

$$= \sum_{n=2}^{\infty} \frac{(n-\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}\left(\sum_{n=1}^{\infty} t_i|a_{n_i}|\right)+\sum_{n=1}^{\infty} \frac{(n+\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}\left(\sum_{n=1}^{\infty} t_i|b_{n_i}|\right)$$

$$\leq \sum_{i=1}^{\infty} t_i\left(\sum_{n=2}^{\infty} \frac{(n-\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}|a_{n_i}| + \sum_{n=1}^{\infty} \frac{(n+\lambda\nu)}{(1-\nu)(2n-1)(n-1)!}\psi_{n-1}|b_{n_i}|\right)$$

$$\leq \sum_{i=1}^{\infty} t_i = 1$$

and hence $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$.

Closure properties of $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ under the generalized Bernardi Libera-Livingston integral operator is defined by

$$F_c(f(z)) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt \quad ; c \in \mathbb{N}.$$

We have established that the result in Theorem 2.2 for functions f in $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ is necessary condition for $F_c(f)$ to be in $\mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$.

Theorem 2.6. *Let $f \in \mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$. Then $F_c(f) \in \mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$.*

Proof. Consider $f \in \mathcal{E}N_{\overline{H}}(\mu,\nu,\lambda)$ and on application of the defined integral operator we can write

$$\begin{aligned} F_c(f(z)) &= \frac{1+c}{z^c}\int_0^z t^{c-1}(h(t)+\overline{g(t)})dt \\ &= \frac{1+c}{z^c}\int_0^z t^{c-1}\left(t+\sum_{n=2}^{\infty}(-1)^n|a_n|t^n + \overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}(-1)^{n-1}|b_n|t^n\right)dt}\right. \\ &= z + \sum_{n=2}^{\infty}\frac{c+1}{c+n}(-1)^n|a_n|z^n + \sum_{n=1}^{\infty}\overline{\frac{c+1}{c+n}(-1)^{n-1}|b_n|z^n}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{n=2}^{\infty}\psi_{n-1}\left(\frac{(n-\lambda\nu)}{1-\nu}\frac{c+1}{c+n}|a_n| + \frac{(n+\lambda\nu)}{1-\nu}\frac{c+1}{c+n}|b_n|\right)\frac{1}{(2n-1)(n-1)!} \\ &\leq \sum_{n=2}^{\infty}\psi_{n-1}\left(\frac{(n-\lambda\nu)}{1-\nu}|a_n| + \frac{(n+\lambda\nu)}{1-\nu}|b_n|\right)\frac{1}{(2n-1)(n-1)!} \\ &\leq 1 - \frac{1+\nu}{1-\nu}|b_1|. \end{aligned}$$

Since $f \in \mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$; by Theorem 2.2, $F_c(f) \in \mathcal{E}N_{\overline{H}}(\mu, \nu, \lambda)$.

3. Concluding Remark

In conclusion, this research provides some useful information on the functions in the class $\mathcal{E}N_H(\mu, \nu, \lambda)$ of harmonic error functions. By defining key properties such as inclusion criteria, extreme points, convexity, distortion bounds, and necessary and sufficient conditions, this work makes a some contribution to the theory of harmonic mappings. However, the study of harmonic error functions is far from complete. Future research could focus on coefficient estimates, radius problems, and the effects of convolution and integral operators on $\mathcal{E}N_H(\mu, \nu, \lambda)$.

Additionally, investigating connections with other function classes and exploring potential generalizations to higher dimensions could further expand the scope of this research. Moreover, examining applications in fields such as geometric function theory, applied mathematics, and image processing could also advance the knowledge in this field. We hope that this paper can be a motivation for further exploration of harmonic error functions and their diverse implications.

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